Math Analysis I
HW 5
Due 03/07/2013

1. If $x, y \in (X, d)$ and $d(x, y) < \epsilon$ for all $\epsilon > 0$, prove that $x = y$

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2.

a) Show that the function

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is a metric on any non-empty set X. (This is called the discrete metric on X.)

b) Show that the following functions define a metric on $\mathbb{R}$.

(a) $d(x, y) = |x - y|$
(b) $d(x, y) = \sqrt{|x - y|}$.

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3.

a) Let $a \in (X, d)$. Prove that if $x_n = a$ for every $n \in \mathbb{N}$, then x_n converges. What does it converge to?

b) Let $\mathbb{R}$ is given with discrete metric. Prove that $x_n \to a$ as $n \to \infty$ if and only if $x_n = a$ for large n.

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4. Let (X, d) be a metric space. Suppose ρ is defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that ρ is also a metric on X. (Note that the new metric ρ is bounded because $\rho(x, y) < 1$ for all $x, y \in X$.)

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5.

a) Show that every convergent sequence in a metric space (X, d) is a Cauchy sequence.

b) Give an example of a metric space(X, d) and a Cauchy sequence $\{x_n\} \subseteq X$ such that $\{x_n\}$ does not converge in X.

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6. The open ball $B(x;r)$ in a metric space (X,d) is defined by

$$B(x;r) := \{y \in X \,:\, d(x,y) < r\}.$$

$B(x;r)$ is called the unit ball if $r = 1$. Draw the unit balls in $\mathbb{R}^2$ centered at $(0,0)$ in $\mathbb{R}^2$ with respect to the metrics

(a) $d_1(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

(b) $d_2(x,y) = |x_1 - y_1| + |x_2 - y_2|$

(c) $d_3(x,y) = \max(|x_1 - y_1|, |x_2 - y_2|)$.

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7.

a) Show that $B(x;r)$ in a Euclidean space is convex.
(Recall that a set X is convex if $x_1, x_2 \in X$ implies that $\alpha x_1 + (1-\alpha)x_2 \in X$where $0 \le \alpha \le 1$).

b) In a metric space (X,d), given a ball $B(x_0;r)$, show that for any $x \in B(x_0;r)$, $B(x;s) \subseteq B(x_0;r)$ for all $0 < s \le r - d(x,x_0)$.

c) Describe a closed ball, open ball, and sphere with center x_0 and radius r in a metric space with the discrete metric.

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8.

a) Let (X,d) be a metric space. Show for all $x,y,z,w \in X$ we have

$$\Big|d(x,z) - d(z,y)\Big| \le d(x,y) \text{ and } \Big|d(x,y) - d(z,w)\Big| \le d(x,z) + d(y,w).$$

b) Let (X,d) be a metric space. Show that if $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences of X, then $\{d(x_n,y_n)\}$ is a Cauchy sequence in $\mathbb{R}$.

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9. Find the limit superior and limit inferior of the sequence $\{x_n\}$, where

a) $x_n = 1 + (-1)^n + \frac{1}{2^n}$.

b) $x_n = 2^n$

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